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From Geometry of Interaction to Denotational Semantics

Esfandiar Haghverdi [1]

School of Informatics & Department of Mathematics
Indiana University Bloomington
Bloomington, Indiana USA

Philip Scott [2,3]

Department of Mathematics and Statistics
University of Ottawa
Ottawa, Ontario Canada

Abstract

We analyze the categorical foundations of Girard's Geometry of Interaction Program for Linear Logic. The motivation for the work comes from the importance of viewing GoI as a new kind of semantics and thus trying to relate it to extant semantics. In an earlier paper we showed that a special case of Abramsky's GoI situations–ones based on Unique Decomposition Categories (UDC's)–*exactly* captures Girard's functional analytic models in his first GoI paper, including Girard's original Execution formula in Hilbert spaces, his notions of orthogonality, types, datum, algorithm, etc. Here we associate to a UDC-based GoI Situation a denotational model (a $*$-autonomous category (without units) with additional exponential structure). We then relate this model to some of the standard GoI models via a fully-faithful embedding into a double-gluing category, thus connecting up GoI with earlier Full Completeness Theorems.

Key words:
Geometry of Interaction, Denotational semantics, Linear logic, *-autonomous categories, Unique decomposition categories.

[1] Email: ehaghver@indiana.edu
[2] Research Supported by an NSERC Discovery Grant
[3] Email: phil@mathstat.uottawa.ca

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1 Introduction

Girard introduced his Geometry of Interaction (GoI) programme in the late 80's in a series of fundamental papers [9,8,10]. Girard's goal in the original GoI was to analyze the dynamics of cut-elimination, using sophisticated mathematical models arising from functional analysis. In the original papers, Girard established a kind of *feedback equation* (known as the *Execution Formula*) which gives an intrinsic measure of "information flow" in the cut-elimination process. In his detailed modelling of proofs, Girard also established that for a large class of types (for example, strong enough to represent System $\mathcal{F}$), the Execution formula is an *invariant* for cut-elimination.

The GoI interpretation was extended to untyped λ-calculus by Danos in [6]. In many subsequent works, Danos and Regnier and coauthors (e.g. [21,7]) further extended the GoI interpretation. They developed a theory of paths in abstract nets (untyped or typed), with detailed comparisons with many λ-calculus notions of path. After Girard's original GoI papers appeared, Joyal, Street, and Verity [19] introduced *traced monoidal categories* (TMC's); balanced monoidal categories with an abstract notion of "trace" or "feedback". These categories have proved useful in many areas ranging from topology and knot theory to theoretical physics and computer science (see Section 2 below). In Linear Logic, the theory of TMC's led to an abstract formalisation of GoI via the notion of *GoI Situation*, introduced by Abramsky in his Siena lecture [2], based on earlier formalizations of GoI in [4], using domain theory. GoI Situations give the essential categorical ingredients of GoI, at least for the multiplicative and exponential (MELL) fragment. Abramsky's programme was sketched in [2] and completed in [11] and [3] (see also Section 2 below).

However two questions remained in [3]:

(i) How to compare the general algebraic framework of a GoI Situation with the actual details of the functional-analytic models introduced by Girard and studied by Danos & Regnier, et al?

(ii) How to compare GoI models with denotational models (of proofs)? In the case of linear logic, this means we want to naturally connect up GoI models with $*$-autonomous categories, with the additional structure to model exponentials.

Re (1), in our first paper [14], we showed how the axiomatics of TMC's in GoI situations (see [3]), when restricted to Unique Decomposition Categories (UDC's) (see below and Section 3), allows us to categorically reconstruct Girard's first model. This model is based on the C^*-algebra of bounded linear operators on the space ℓ^2 of square summable sequences [9]. Our categorical approach permits an elegant derivation of Girard's original execution formula in his model, explicates his notion of type, datum, and algorithm and clarifies the role of the later theory of TMCs in Girard's original proofs.

Re (2), traditional semantics models cut-elimination by static equalities.

This means that if Π, Π' are proofs of a sequent $\Gamma \vdash A$ and if we have a reduction $\Pi \succ \Pi'$ by cut-elimination, then in any categorical model their interpretations $[\![-]\!]$ denote equal morphisms, i.e. $[\![\Pi]\!] = [\![\Pi']\!] : [\![\Gamma]\!] \to [\![A]\!]$. The goal of GoI is to provide a mathematical model of the dynamics of cut-elimination, independent of the syntax.

In this paper, as in our paper [14], we restrict the abstract TMC's to a useful subclass: traced unique decomposition categories (Traced UDC's) with standard trace [11,12]. These are symmetric monoidal categories whose homsets are enriched with certain infinitary partial sums, thus allowing us to consider morphisms as matrices and the execution formula as an infinite sum. Such categories are inspired from early categorical analyses of programming languages by Elgot, Arbib and Manes, et al. ([20]).

We start with a UDC-GoI Situation and construct a denotational model for MELL without units. It is a $*$-autonomous category without units, together with an endofunctor satisfying certain axioms. We show this denotational category may be fully and faithfully embedded in a double-gluing category [11,18,24] built via the $\mathcal{G}$ construction of Abramsky (shown to be isomorphic to JSV's *Int* construction in [11].) This not only connects up our theory with the known denotational models already studied in [3,11,12] but also connects with fully complete MLL models arising from GoI [11,13]. The rest of the paper is organized as follows: In Section 2 we recall the necessary definitions, namely traced symmetric monoidal categories and GoI Situations, following [11,3]. In Section 3 we recall the definition of a unique decomposition category and give some examples. Section 4 briefly recalls the GoI interpretation for MELL formulas and proofs taken directly from [14]. Section 5, is the main part of the paper where we explain and detail the construction of a $*$-autonomous category from a UDC-GoI Situation that we call *the orthogonality construction*. In Section 6 we relate this latter category to double-gluing categories familiar from work in Full Completeness. Finally in section 7 we conclude by discussing related and future work.

2 Traced Monoidal Categories and GoI Situation

Joyal, Street and Verity [19] introduced the notion of an abstract trace on a balanced monoidal category (a monoidal category with braiding and twist.) This trace can be interpreted in various contexts where it could be called feedback, parametrized fixed-point, Markov trace or braid closure. These categories have their origins in the analysis of braided tensor categories and in knot theory. However the special case of traced *symmetric* monoidal categories have been particularly useful in some areas of theoretical computer science, for example in cyclic lambda calculi [15], semantics of asynchronous networks [22], full completeness theorems for multiplicative linear logic via GoI [11,13], analysis of finite state machines [17], relational dataflow [16], and they independently arose in Stefanescu's work in network algebra [23].

In what follows we only consider *symmetric* monoidal categories.

Definition 2.1 A *traced symmetric monoidal category* (TMC) is a symmetric monoidal category $(\mathbb{C}, \otimes, I, s)$ with a family of functions $Tr^U_{X,Y} : \mathbb{C}(X \otimes U, Y \otimes U) \to \mathbb{C}(X, Y)$ called a *trace*, subject to the following axioms:

- **Natural** in X, $Tr^U_{X,Y}(f)g = Tr^U_{X',Y}(f(g \otimes 1_U))$ where $f : X \otimes U \to Y \otimes U$, $g : X' \to X$,
- **Natural** in Y, $gTr^U_{X,Y}(f) = Tr^U_{X,Y'}((g \otimes 1_U)f)$ where $f : X \otimes U \to Y \otimes U$, $g : Y \to Y'$,
- **Dinatural** in U, $Tr^U_{X,Y}((1_Y \otimes g)f) = Tr^{U'}_{X,Y}(f(1_X \otimes g))$ where $f : X \otimes U \to Y \otimes U'$, $g : U' \to U$,
- **Vanishing (I,II)**, $Tr^I_{X,Y}(f) = f$ and $Tr^{U \otimes V}_{X,Y}(g) = Tr^U_{X,Y}(Tr^V_{X \otimes U, Y \otimes U}(g))$ for $f : X \otimes I \to Y \otimes I$ and $g : X \otimes U \otimes V \to Y \otimes U \otimes V$,
- **Superposing**,
$$Tr^U_{X,Y}(f) \otimes g = Tr^U_{X \otimes W, Y \otimes Z}((1_Y \otimes s_{U,Z})(f \otimes g)(1_X \otimes s_{W,U}))$$
for $f : X \otimes U \to Y \otimes U$ and $g : W \to Z$,
- **Yanking**, $Tr^U_{U,U}(s_{U,U}) = 1_U$.

TMC's admit a geometric diagram calculus that can be found in the references (e.g. [11,3,19]).

Joyal, Street, and Verity [19] also introduced the *Int* construction on traced symmetric monoidal categories $\mathbb{C}$; $Int(\mathbb{C})$ is a kind of "free compact closure" of the category $\mathbb{C}$. $Int(\mathbb{C})$ is used in [19] to give a 2-categorical structure theorem for TMC's. $Int(\mathbb{C})$ isolates the key properties of Girard's GoI for the multiplicative connectives; for example composition in $Int(\mathbb{C})$ uses a version of Girard's Execution Formula applied to the GoI interpretation of the cut rule. Abramsky [1] independently introduced the $\mathcal{G}$ construction which associates a compact closed category to a traced *symmetric* monoidal one. In [11] the two constructions are shown to yield isomorphic compact closed categories starting with the same TMC. There are two problems: first, these (isomorphic) constructions only yield "degenerate" compact closed models for MLL (so tensor = par). Second is the problem of how to extend this to the exponential connectives.

Re the second problem, in the Abramsky program (see [3]) this is achieved by adding certain additional structure to a traced symmetric monoidal category. This structure involves a monoidal endofunctor T, a reflexive object U, and appropriate monoidal retractions, as introduced below, to yield a *GoI situation*. It was shown in [3] that GoI situations endow the monoid $\mathbb{C}(U, U)$ with the structure of a *linear combinatory algebra*. Such combinatory algebras capture the appropriate computational meaning of the exponentials in linear logic and model a Hilbert-style presentation of MELL.

Re the first problem above, it is possible to construct a non-compact $*$-autonomous category from a given compact closed one using the double glueing construction of Hyland and Tan [24,18] on top of the compact closed category. On the other hand, in this paper we start with a UDC-GoI Situation and directly construct a $*$-autonomous category and an endofunctor ! on it to get a denotational model for MELL. Here not only is the construction direct but more importantly it exploits the structure of a GoI Situation. The *Int* (or $\mathcal{G}$) constructions do not take into account either the rôle of the reflexive object U or the orthogonality relation defined on $\mathbb{C}(U,U)$ (see the definition below). We believe that the latter two are among the most important and interesting ingredients of the GoI interpretation.

Definition 2.2 A *GoI Situation* is a triple $(\mathbb{C}, T, U)$ where:

(i) $\mathbb{C}$ is a traced symmetric monoidal category

(ii) $T : \mathbb{C} \to \mathbb{C}$ is a traced symmetric monoidal functor with the following retractions (note that the retraction pairs are monoidal natural transformations):
  - (a) $TT \lhd T \ (e, e')$ (Comultiplication)
  - (b) $Id \lhd T \ (d, d')$ (Dereliction)
  - (c) $T \otimes T \lhd T \ (c, c')$ (Contraction)
  - (d) $\mathcal{K}_I \lhd T \ (w, w')$ (Weakening). Here $\mathcal{K}_I$ is the constant I functor.

(iii) U is an object of $\mathbb{C}$, called *a reflexive object*, with the specified retractions: (a) $U \otimes U \lhd U \ (j, k)$, (b) $I \lhd U \ (m, n)$, and (c) $TU \lhd U \ (u, v)$.

For examples of GoI Situations see Section 3. For our models of linear logic, we will take the following definitions.

Definition 2.3 A symmetric monoidal category $(\mathbb{C}, \otimes, I, s)$ is a $*$-*autonomous category* if there exists a full and faithful functor $(-)^{\perp} : \mathbb{C}^{op} \to \mathbb{C}$ such that there exists an isomorphism $\mathbb{C}(A \otimes B, C^{\perp}) \to \mathbb{C}(A, (B \otimes C)^{\perp})$ natural in A, B and C.

The models of multiplicative linear logic (MLL) are $*$-autonomous categories. For the multiplicative and exponential fragment (MELL), we assume:

Definition 2.4 A denotational model of MELL consists of the following data:

(i) A $*$-autonomous category $(\mathbb{C}, \otimes, I, s, (-)^{\perp})$,

(ii) A symmetric monoidal functor $(!, \varphi, \varphi_I) : \mathbb{C} \to \mathbb{C}$.

(iii) Monoidal natural transformations:
  - (a) $der : ! \Longrightarrow Id$
  - (b) $\delta : ! \Longrightarrow !!$
  - (c) $weak : ! \Longrightarrow \mathcal{K}_I$ where $\mathcal{K}_I$ is the constant I functor
  - (d) $con : ! \Longrightarrow ! \otimes !$

  such that
  - $(!, der, \delta)$ is a comonad.
  - for each object A, the triple $(!A, weak_A, con_A)$ is a commutative comonoid.

- for each object A, the maps $weak_A$ and con_A are maps of coalgebras.
- for each object A, the map δ_A is a map of commutative comonoids.

Finally, we remark that there are two "styles" of GoI Situations in the concrete models studied in [3]: *Sum style* and *Product style*. These are determined by the form of the tensor in the underlying TMC. Roughly, in sum style, the tensor $\otimes$ is given by a disjoint union on objects; in product style, it is more like a cartesian product. We shall exclusively consider Sum style models here, corresponding to Girard's GoI 1. Sum style GoI admits a semantics based on "particles flowing through a network" [3,11].

3 Unique Decomposition Categories

We consider monoidal categories whose homsets allow the formation of certain infinite sums. These are monoidal categories enriched in Σ-monoids (see below). In the case where the tensor is coproduct and Σ-monoids satisfy an additional condition, such categories include the *partially additive and iterative* categories used in the early categorical analyses of flow charts and programming languages by Bainbridge, Elgot, Arbib and Manes, et. al. (e.g. [20]).

Definition 3.1 A Σ-*monoid* consists of a pair (M, Σ) where M is a nonempty set and Σ is a partial operation on the countable families in M (we say that $\{x_i\}_{i\in I}$ is *summable* if $\sum_{i\in I} x_i$ is defined), subject to the following axioms:

(i) *Partition-Associativity Axiom.* If $\{x_i\}_{i\in I}$ is a countable family and if $\{I_j\}_{j\in J}$ is a (countable) partition of I , then $\{x_i\}_{i\in I}$ is summable if and only if $\{x_i\}_{i\in I_j}$ is summable for every $j \in J$ and $\sum_{i\in I_j} x_i$ is summable for $j \in J$. In that case, $\sum_{i\in I} x_i = \sum_{j\in J}(\sum_{i\in I_j} x_i)$

(ii) *Unary Sum Axiom.* Any family $\{x_i\}_{i\in I}$ in which I is a singleton is summable and $\sum_{i\in I} x_i = x_j$ if $I = \{j\}$.

Σ-monoids form a symmetric monoidal category (with product as tensor), called Σ**Mon**. A Σ**Mon**-category $\mathbb{C}$ is a category enriched in Σ**Mon**; i.e. the homsets are enriched with a partial infinitary sum compatible with composition. Note that such categories have non-empty homsets and automatically have zero morphisms, namely $0_{XY} : X \to Y = \sum_{i\in\emptyset} f_i$ for $f_i \in \mathbb{C}(X, Y)$. For details see [20,11].

Definition 3.2 A *unique decomposition category* (UDC) $\mathbb{C}$ is a symmetric monoidal Σ**Mon**-category which satisfies the following axiom:

(A) For all $j \in I$ there are morphisms called *quasi injections*: $\iota_j : X_j \to \otimes_I X_i$, and *quasi projections*: $\rho_j : \otimes_I X_i \to X_j$, such that

1. $\rho_k \iota_j = 1_{X_j}$ if $j = k$ and $0_{X_j X_k}$ otherwise.
2. $\sum_{i\in I} \iota_i \rho_i = 1_{\otimes_I X_i}$.

Proposition 3.3 (Matricial Representation) *Given $f : \otimes_J X_j \to \otimes_I Y_i$ in a UDC with $|I| = m$ and $|J| = n$, there exists a unique family $\{f_{ij}\}_{i\in I, j\in J} : X_j \to Y_i$ with $f = \sum_{i\in I, j\in J} \iota_i f_{ij} \rho_j$, namely, $f_{ij} = \rho_i f \iota_j$.*

Thus every morphism $f : \otimes_J X_j \to \otimes_I Y_i$ in a UDC can be represented by a matrix; for example f above (with $|I| = m$ and $|J| = n$) is represented by the $m \times n$ matrix $[f_{ij}]$. Composition of morphisms in a UDC then corresponds to matrix multiplication.

Proposition 3.4 (Standard Trace Formula) *Let $\mathbb{C}$ be a unique decomposition category such that for every X, Y, U and $f : X \otimes U \to Y \otimes U$, the sum $f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$ exists, where f_{ij} are the components of f* [4]*. Then, $\mathbb{C}$ is traced and $Tr^U_{X,Y}(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}$.*

The trace formula above is called the standard trace, and a UDC with such a trace is called a *traced UDC with standard trace.* Note that a UDC can be traced with a trace different from the standard one. In this paper all traced UDCs are the ones with the standard trace.

The following examples have standard trace, as above.

Examples 3.5 (Traced UDC's) (For details see [3,11,13]).

(i) Any partially-additive category (see [20]). This includes:
Rel$_+$ (sets and relations). Here $\otimes = \uplus$ (disjoint union, which is a biproduct). In **Rel$_+$**, *all* countable families are summable, and $\sum_{i\in I} R_i = \cup_i R_i$.
Pfn (sets and partial functions), with $\otimes = \uplus$. Define a countable family of partial functions $\{f_i\}_{i\in I}$ to be summable iff they have pairwise disjoint domains. Then $(\sum_{i\in I} f_i)(x) = f_j(x)$ iff $x \in Dom(f_j)$, for some $j \in I$, otherwise undefined.
SRel, the category of stochastic relations. Here the objects are measurable spaces $(X, \mathcal{F}_X)$ and maps $f : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ are stochastic kernels, i.e. $f : X \times \mathcal{F}_Y \to [0, 1]$ such that $f(x, .)$ is a subprobability measure and $f(., B)$ is a bounded measurable function, for all $x \in X$ and $B \in \mathcal{F}_Y$. Composition $g \circ f(x, C) = \int_Y g(y, C) f(x, dy)$, where $f(x, .)$ is the measure for integration. This category has finite and countable coproducts (which form the tensor). A family $\{f_i\}_{i\in I}$ is summable iff $\sum_{i\in I} f_i(x, Y) \le 1$ for all $x \in X$.

(ii) **PInj** (sets and partial injective functions). Here $\otimes = \uplus$; this is not a coproduct, indeed **PInj** does not have coproducts. The UDC structure is $X_j \xrightarrow{\iota_j} \uplus_{i\in I} X_i$ with $\iota_j(x) = (x, j)$, and $\uplus_{i\in I} X_i \xrightarrow{\rho_j} X_j$ with $\rho_j(x, j) = x$ and $\rho_j(x, i)$ undefined for $i \ne j$. Summable families are as in **Pfn** but with disjoint domains & codomains.

(iii) **Hilb$_2$**. Consider the category **Hilb** of Hilbert spaces and linear contractions (norm ≤ 1). Barr [5] defined a contravariant faithful functor

[4] Here $X_1 = X, Y_1 = Y, X_2 = Y_2 = U$. So $f_{11} : X \to Y, f_{12} : U \to Y$, etc.

ℓ_2 :$\mathbf{PInj}^{op}$ $\rightarrow$**Hilb** by: for a set X, $\ell_2(X)$ is the set of all complex valued functions a on X for which the (unordered) sum $\sum_{x\in X}|a(x)|^2$ is finite. $\ell_2(X)$ is a Hilbert space with norm given by $||a|| = (\sum_{x\in X}|a(x)|^2)^{1/2}$ and inner product given by $< a,b >= \sum_{x\in X} a(x)\overline{b(x)}$ for $a,b\in\ell_2(X)$. Given $f : X \to Y$ in **PInj**, define $\ell_2(f) : \ell_2(Y) \to \ell_2(X)$ by $\ell_2(f)(b)(x) = b(f(x))$ if $x \in Dom(f)$ and $= 0$, otherwise.

This gives a correspondence between partial injective functions and partial isometries on Hilbert spaces (see also [10,1].) Let $\mathbf{Hilb}_2 = \ell_2[\mathbf{PInj}]$. Its objects are $\ell_2(X)$ for a set X and morphisms $u : \ell_2(X) \to \ell_2(Y)$ are of the form $\ell_2(f)$ for some partial injective function $Y \xrightarrow{f} X$. Hence, $\mathbf{Hilb}_2$ is a nonfull subcategory of **Hilb**. It forms a traced UDC with respect to $\oplus$, where $\ell_2(X) \oplus \ell_2(Y) \cong \ell_2(X \uplus Y)$ is a tensor product in $\mathbf{Hilb}_2$ (but is a biproduct in **Hilb**) with the necessary structure induced by ℓ_2 from **PInj**.

The above examples yield GoI situations $(\mathbb{C}, T, U)$ with T an additive functor ([3,11]):

- $(\mathbf{Rel}_+, T, \mathbb{N})$, $(\mathbf{Pfn}, T, \mathbb{N})$, and $(\mathbf{PInj}, T, \mathbb{N})$ with $T = \mathbb{N} \times -$.
- $(\mathbf{Hilb}_2, T, \ell^2)$ with $T = \ell^2 \otimes -$, where $\ell^2 = \ell_2(\mathbb{N})$.
- $(\mathbf{SRel}, T, \mathbb{N}^{\mathbb{N}})$, where $T(X, \mathcal{F}_X) = (\mathbb{N}\times X, \mathcal{F}_{\mathbb{N}\times X})$, where $\mathcal{F}_{\mathbb{N}\times X}$ is the σ-field generated by $\biguplus_{\mathbb{N}} X$.

4 The GoI Interpretation for MELL

We remind the reader of definitions and results pertaining to the GoI interpretation of MELL that we shall be using in the sequel. These are crucial for a proper understanding of the results in this paper. For more details see [9] for the original definitions and our [14] for the categorical version. We follow [14].

In the sequel $\mathbb{C}$ is a traced UDC with standard trace, T an additive endofunctor and U an object of $\mathbb{C}$, such that $(\mathbb{C}, T, U)$ forms a GoI Situation. We interpret proofs in the homset $\mathbb{C}(U,U)$ and formulas (= types) are interpreted as certain subsets of $\mathbb{C}(U,U)$.

Convention: We write 1_Γ instead of 1_{U^n}, where $|\Gamma| = n$ and where U^n denotes the n-fold tensor product of U with itself. The retraction pairs are fixed once and for all using the names in Definition 2.2. j_1, j_2 and k_1, k_2 denote the components of j and k respectively. If A and B are square matrices of size $n \times n$ and $m \times m$, resp., then $A \otimes B$ denotes the $n + m \times n + m$ block matrix with A and B on the "main diagonal" and the rest zeros.

Definition 4.1 Let $f, g \in \mathbb{C}(U,U)$. We say that f is *nilpotent* if $f^k = 0$ for some $k \geq 1$. We say f is *orthogonal* to g, denoted $f \perp g$ if gf is nilpotent. Orthogonality is a symmetric relation, well-defined since 0_{UU} exists. Also,

$0 \perp f$ for all $f \in \mathbb{C}(U, U)$.

Given a subset X of $\mathbb{C}(U, U)$, we define

$$X^{\perp} = \{f \in \mathbb{C}(U, U) | \forall g(g \in X \Rightarrow f \perp g)\}$$

A *type* is any subset X of $\mathbb{C}(U, U)$ such that $X = X^{\perp\perp}$. Note that types are inhabited, since 0_{UU} belongs to every type.

Definition 4.2 Consider a GoI Situation $(\mathbb{C}, T, U)$. Let A be an MELL formula. We define *the GoI interpretation of* A, denoted θA, inductively as follows:

(i) If $A \equiv \alpha$ that is A is an atom, then $\theta A = X$ an arbitrary type.

(ii) If $A \equiv \alpha^{\perp}$, $\theta A = X^{\perp}$, where $\theta\alpha = X$ is given by assumption.

(iii) If $A \equiv B \otimes C$, $\theta A = Y^{\perp\perp}$ where $Y = \{j_1 a k_1 + j_2 b k_2 | a \in \theta B, b \in \theta C\}$.

(iv) If $A \equiv B \invamp C$, $\theta A = Y^{\perp}$, where $Y = \{j_1 a k_1 + j_2 b k_2 | a \in (\theta B)^{\perp}, b \in (\theta C)^{\perp}\}$.

(v) If $A \equiv !B$, $\theta A = Y^{\perp\perp}$, where $Y = \{uT(a)v | a \in \theta B\}$.

(vi) If $A \equiv ?B$, $\theta A = Y^{\perp}$, where $Y = \{uT(a)v | a \in (\theta B)^{\perp}\}$.

It is an easy consequence of the definition that $(\theta A)^{\perp} = \theta A^{\perp}$ for any formula A.

Every MELL sequent will be of the form $\vdash [\Delta], \Gamma$ where Γ is a sequence of formulas and Δ is a sequence of cut formulas that have already been made in the proof of $\vdash \Gamma$ (e.g. $A, A^{\perp}, B, B^{\perp}$). This is used to keep track of the cuts. Suppose $|\Gamma| = n$, $|\Delta| = 2m$ formulas. Then *the GoI interpretation of a proof* Π *of* $\vdash [\Delta], \Gamma$ is represented by a pair $([\![\Pi]\!], \sigma)$, where $[\![\Pi]\!] \in \mathbb{C}(U^{n+2m}, U^{n+2m})$ and the morphism $\sigma : U^{2m} \to U^{2m}$ which models the cuts Δ in $\vdash [\Delta], \Gamma$ is defined as $\sigma = s \otimes \cdots \otimes s$ (m-copies) where s is the symmetry map, the 2×2 antidiagonal matrix $[a_{ij}]$, where $a_{12} = a_{21} = 1; a_{11} = a_{22} = 0$. In the case where Δ is empty, the proof is cut-free, we define $\sigma : I \to I$ to be $1_I = 0_{II}$. Note that $U^0 = I$ where I is the unit of the tensor in the category $\mathbb{C}$. It is much more convenient to work in $\mathbb{C}(U^{n+2m}, U^{n+2m})$ (matrices on $\mathbb{C}(U, U)$), although by the retractions we can equally work in $\mathbb{C}(U, U)$.

Let Π be a proof of $\vdash [\Delta], \Gamma$. We define the GoI interpretation of Π, denoted by $[\![\Pi]\!]$, by induction on the length of the proof as follows. For lack of room, we only give three cases and refer to [14] for details and the associated (block) matrix representation. Pictorially picture Π as an I/O box, with $n + 2m$ wires (labelled by the formulas in Γ, Δ) coming in and out. The wires are the interface.

(i) Π is an *axiom* $\vdash A, A^{\perp}$, then $m = 0$, $n = 2$ and $[\![\Pi]\!] = s$.

(ii) Π is obtained using the *cut* rule on Π' and Π'' that is

$$\dfrac{\begin{array}{cc}\Pi' & \Pi'' \\ \vdots & \vdots \\ \vdash [\Delta'], \Gamma', A & \vdash [\Delta''], A^{\perp}, \Gamma''\end{array}}{\vdash [\Delta', \Delta'', A, A^{\perp}], \Gamma', \Gamma''} \ (cut)$$

Define $[\![\Pi]\!]$ as follows: $[\![\Pi]\!] = \tau^{-1}([\![\Pi']\!] \otimes [\![\Pi'']\!])\tau$, where τ is a permutation "rearranging the interface" (pictorially, stack one I/O box on top of the other; τ permutes the wires to put all the cut formulas adjacent.)

A similar interpretation holds for the times and par rules.

(iii) Π is obtained from Π' by the *contraction* rule, that is Π is of the form

$$\dfrac{\begin{array}{c}\Pi' \\ \vdots \\ \vdash [\Delta], \Gamma', ?A, ?A\end{array}}{\vdash [\Delta], \Gamma', ?A} \ (contraction)$$

Then $[\![\Pi]\!] = (1_{\Gamma'} \otimes uc_U(v \otimes v) \otimes 1_{\Delta})[\![\Pi']\!](1_{\Gamma'} \otimes (u \otimes u)c'_U v \otimes 1_{\Delta})$, where $T \otimes T \lhd T\,(c, c')$.

Definition 4.3 (Girard[9]) We are in a traced UDC-GoI situation. Let $\Gamma = A_1, \cdots, A_n$. A *datum* of type $\theta\Gamma$ is a morphism $M : U^n \to U^n$ such that for any $\beta_1 \in \theta(A_1^{\perp}), \cdots, \beta_n \in \theta(A_n^{\perp})$, $(\beta_1 \otimes \cdots \otimes \beta_n)M$ is nilpotent. An *algorithm* of type $\theta\Gamma$ is a morphism $M : U^{n+2m} \to U^{n+2m}$ for some non-negative integer m such that for $\sigma : U^{2m} \to U^{2m}$ defined in the usual way, $EX(M, \sigma)$ is a finite sum and a datum of type $\theta\Gamma$, where

$$EX(M, \sigma) = Tr^{U^{2m}}_{U^n, U^n}((1_{U^n} \otimes \sigma)M)$$

is known as the *Execution Formula for M*.

Lemma 4.4 *Let $M : U^n \to U^n$ and $a : U \to U$. Define $CUT(a, M) = (a \otimes 1_{U^{n-1}})M : U^n \to U^n$. Then $M = [m_{ij}]$ is a datum of type $\theta(A, \Gamma)$ iff for any $a \in \theta A^{\perp}$, am_{11} is nilpotent and the morphism $ex(CUT(a, M)) = Tr^A(s^{-1}_{\Gamma,A} CUT(a, M) s_{\Gamma,A}) \in \theta(\Gamma)$. Here $s_{\Gamma,A}$ is the symmetry morphism.*

The following is due to Girard [9], and proved categorically in our [14]:

Theorem 4.5 (Girard) *Let Π be a proof of a sequent $\vdash [\Delta], \Gamma$ in MELL with $|\Delta| = 2m$ and $|\Gamma| = n$. Then*

(i) $[\![\Pi]\!]$ is an algorithm of type $\theta\Gamma$, in particular $EX([\![\Pi]\!], \sigma)$ is a finite sum.

(ii) If Π reduces to Π' by any sequence of cut-eliminations and "?" does not occur in Γ, then $EX([\![\Pi]\!], \sigma) = EX([\![\Pi']\!], \tau)$. So $EX([\![\Pi]\!], \sigma)$ is an invariant of cut-elimination.

(iii) If Π' is the normal form of Π under cut-elimination, then $EX([\![\Pi]\!], \sigma) = [\![\Pi']\!]$.

(iv) *In* $\mathbf{Hilb}_2$*,* $EX([\![\Pi]\!],\sigma) = ((1-\tilde{\sigma}^2)\sum_{n=0}^{\infty}[\![\Pi]\!](\tilde{\sigma}([\![\Pi]\!]))^n(1-\tilde{\sigma}^2))_{n\times n}$*. Here* $(A)_{n\times n}$ *denotes the* $n \times n$ *submatrix of the matrix* A *consisting of the first* n *rows and the first* n *columns of* A *and* $\tilde{\sigma}$ *is the* $n+2m$ *square matrix* $\mathbf{0}_n \otimes \sigma$*, where* $\mathbf{0}_n$ *is* $n \times n$ *zero matrix and* σ *is the* $2m \times 2m$ *square matrix associated with the cuts* Δ*.*

5 Orthogonality Construction

We explain the orthogonality construction which given a UDC-GoI Situation yields a $*$-autonomous category without units and an endofunctor on it forming a denotational model of MELL without units. In the sequel we have omitted many of the routine and tedious verifications (these will appear in the long version). However, in each and every case we explain the reasoning behind the definition, so that the reader would not get the wrong impression that the definitions are somehow arbitrary. The general intuition behind this construction is to use the GoI interpretation for formulae to define the objects and to use the GoI interpretation of a cut-free proof of $\vdash A^{\perp}, B$ to define a morphism $f : A \to B$. In Girard's terminology above, kfj is a *datum of type* $\theta(A^{\perp}, B)$.

Given a UDC-GoI Situation $(\mathbb{C}, T, U)$, we define the category $\mathcal{O}(\mathbb{C})$ as follows:

• **Objects:** An object A is a subset of $\mathbb{C}(U,U)$ such that $A^{\perp\perp} = A$. Recall that these are the types (GoI interpretation for formulae) defined in the previous section.

• **Arrows:** An arrow $f : A \to B$ is a morphism f in $\mathbb{C}(U,U)$ such that for every $a \in A$, $f{\cdot}a =_{def} Tr^U_{U,U}(s_{U,U}(a \otimes 1_U)(kfj)s_{U,U})$ is in B.

Note that as we are working in a traced UDC, we have

$$f{\cdot}a = k_2fj_2 + \sum_{n\geq 0} k_2fj_1(ak_1fj_1)^n ak_1fj_2.$$

The intuition is that we think of $f : A \to B$ as the interpretation of a cut-free proof of $\vdash A^{\perp}, B$, or in other words we think of kfj as a datum of type $\theta(A^{\perp}, B)$, see Definition 4.3 and Theorem 4.5 above. As a matter of fact for those familiar with Girard's work, this definition is exactly $ex(CUT(a, kfj))$ formulated in terms of categorical trace explained in [14] and cited in Lemma 4.4 in the previous Section.

• **Identity:** The identity morphism on A, denoted 1_A, is given by $js_{U,U}k \in \mathbb{C}(U,U)$. Note that for any $a \in A$, $1_A{\cdot}a = Tr^U_{U,U}(s_{U,U}(a \otimes 1_U)(kjs_{U,U}kj)s_{U,U}) = Tr^U_{U,U}(s_{U,U}(a \otimes 1_U)s_{U,U}s_{U,U}) = a \in A$. The latter equality is known as *generalized yanking* in TMC's (see [11,12].) The intuition is that we use the GoI interpretation of the cut-free proof of $\vdash A^{\perp}, A$.

• **Composition:** Composition is defined as follows: given $f : A \to B$ and

$g : B \to C$ in $\mathcal{O}(\mathbb{C})$,

$$gf = j\,Tr^{U\otimes U}_{U\otimes U, U\otimes U}((1\otimes 1\otimes s)\tau^{-1}(kfj\otimes kgj)\tau)k.$$

where $\tau = (1\otimes 1\otimes s)(1\otimes s\otimes 1)$. The motivation comes from the GoI interpretation as follows: we take the cut-free proofs of $\vdash A^\perp, B$ and $\vdash B^\perp, C$ and apply the GoI interpretation for the cut rule to these two proofs, hence we get a proof with cuts in it, namely we get a proof of $\vdash [B, B^\perp], A^\perp, C$. But we need a cut-free proof so we apply the execution formula to this latter proof to get the GoI interpretation of a cut-free proof of $\vdash A^\perp, C$. Our definition precisely reflects these operations.

Note that $gf \in \mathbb{C}(U, U)$. As we are working in a traced UDC we have

$$gf = j\,Tr \begin{bmatrix} k_1fj_1 & 0 & k_1fj_2 & 0 \\ 0 & k_2gj_2 & 0 & k_2gj_1 \\ 0 & k_1gj_2 & 0 & k_1gj_1 \\ k_2fj_1 & 0 & k_2fj_2 & 0 \end{bmatrix} k$$

If $a \in A$ then $(gf)\bullet a \in C$; this follows from the construction and Theorem 4.5 which implies that $k(gf)j$ is a datum of type $\theta(A^\perp, C)$. Hence gf is a well-defined morphism in $\mathcal{O}(\mathbb{C})$.

Note that this is essentially the same as the formula for composition in $\mathcal{G}(\mathbb{C})$ (called *symmetric feedback* in [1]), see [11,1], and of course this is no surprise as the definition of composition in $\mathcal{G}(\mathbb{C})$ is also motivated by the execution formula applied to the cut of two proofs. This is discussed further in Section 6 below.

Proposition 5.1 *Let $(\mathbb{C}, T, U)$ be a UDC-GoI Situation with the additional requirement that $U\otimes U \cong U$ (j, k). Then, $\mathcal{O}(\mathbb{C})$ is a category.*

Proof. As mentioned above the composition and identity morphisms are similar to those in $\mathcal{G}(\mathbb{C})$ and hence the associativity and unit equations hold true. However just to illustrate, let's look at $1_B f = f$ for $f : A \to B$.

$$1_B f = j\,Tr \begin{bmatrix} k_1fj_1 & 0 & k_1fj_2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ k_2fj_1 & 0 & k_2fj_2 & 0 \end{bmatrix} k = j\left(\begin{bmatrix} k_1fj_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_1fj_2 \\ k_2fj_1 & k_2fj_2 \end{bmatrix}\right) k = f$$

□

We next define the $*$-autonomous structure. Given A and B objects in $\mathcal{O}(\mathbb{C})$, define:

• **Tensor**: $A \otimes B = \{j_1 a k_1 + j_2 b k_2 | a \in A, b \in B\}^{\perp\perp}$. Given $f : A \to B$ and $g : A' \to B'$ we define

$$f \otimes g = j(j \otimes j)(1 \otimes s \otimes 1)(kfj \otimes kgj)(1 \otimes s \otimes 1)(k \otimes k)k.$$

Notice that the tensor product used on the right hand side is the one in $\mathbb{C}$. Here is the proof that motivates this definition (ignoring the exchange rule hereafter):

$$\dfrac{\dfrac{\vdash A^{\perp}, B \quad , \vdash A'^{\perp}, B'}{\vdash A^{\perp}, A'^{\perp}, B \otimes B'}\, times}{\vdash A^{\perp} \parr A'^{\perp}, B \otimes B'}\, par$$

• **"Tensor Unit"**: The candidate for the unit of tensor is given by $I = \{1_U\}^{\perp\perp}$. Below we shall show that it falls short; instead we get $A \lhd A \otimes I$ for every object A.

• **Symmetry**: The symmetry $s_{A,B} : A \otimes B \to B \otimes A$ is defined as

$$s_{A,B} = j(j \otimes j)(s \otimes 1 \otimes 1)(1 \otimes s \otimes 1)(s \otimes s)(1 \otimes s \otimes 1)(s \otimes 1 \otimes 1)(k \otimes k)k.$$

Here is the proof that motivates this definition:

$$\dfrac{\dfrac{\dfrac{\vdash B^{\perp}, B \quad , \vdash A^{\perp}, A}{\vdash B^{\perp}, A^{\perp}, B \otimes A}\, times}{\vdash A^{\perp}, B^{\perp}, B \otimes A}\, exchange}{\vdash A^{\perp} \parr B^{\perp}, B \otimes A}\, par$$

• **Duality**: Given A define

$$A^{\perp} = \{f \in \mathbb{C}(U, U) | f \perp g, \text{for all } g \in A\}.$$

Note that $A = A^{\perp\perp}$ by definition of objects in $\mathcal{O}(\mathbb{C})$.

• **Par product**: Given A and B objects of $\mathcal{O}(\mathbb{C})$ we define

$$A \parr B = \{j_1 a k_1 + j_2 b k_2 | a \in A^{\perp}, b \in B^{\perp}\}^{\perp}.$$

• **"Par Unit"** The candidate for unit of par is of course $\perp = \{1_U\}^{\perp}$, however as pointed out above for the case of tensor, $\perp$ fails to be the unit of par.

Theorem 5.2 *Let $(\mathbb{C}, T, U)$ be a UDC-GoI Situation with the additional requirement that $U \otimes U \cong U\,(j, k)$. Then $\mathcal{O}(\mathbb{C})$ is a $*$-autonomous category without units.*

Proof. First we show that tensor is a bifunctor: note that $1_A \otimes 1_B = j \begin{bmatrix} 0 & j_1 k_1 + j_2 k_2 \\ j_1 k_1 + j_2 k_2 & 0 \end{bmatrix} k = jsk = 1_{A \otimes B}$. This uses the fact that $jk =$

1_U. It can also be shown that $f'f \otimes g'g = (f' \otimes g')(f \otimes g)$, for f, g, f', g' of appropriate types using similar matrix calculations.

We define the structure morphisms as follows: $\rho_A : A \otimes I \to A$ is defined by $\rho_A = j(j_1 \otimes k_1)sk = j_1^2 k_2 + j_2 k_1^2$ and $\rho'_A : A \to A \otimes I = j(k_1 \otimes j_1)sk = j_1 k_1 k_2 + j_2 j_1 k_1$. $\lambda_A : I \otimes A \to A = j(j_2 \otimes k_2)sk$, and $\lambda'_A : A \to I \otimes A = j(k_2 \otimes j_2)sk$. Finally $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ is defined as

$$\alpha_{A,B,C} = j(j \otimes 1)(1 \otimes j \otimes j)(1_{U^2} \otimes s \otimes 1)(1_{U^2} \otimes j \otimes 1_{U^2})(1 \otimes s \otimes 1_{U^3})(s \otimes s \otimes s)$$

$$(1 \otimes s \otimes 1_{U^3})(1_{U^2} \otimes k \otimes 1_{U^2})(1_{U^2} \otimes s \otimes 1)(1 \otimes k \otimes k)(k \otimes 1)k.$$

This is motivated by the proof below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash A^\perp, A \quad \vdash B^\perp, B}{\vdash A^\perp, B^\perp, (A \otimes B) \quad \vdash C^\perp, C}}{\vdash A^\perp, B^\perp, C^\perp, (A \otimes B) \otimes C}}{\vdash A^\perp, B^\perp ⅋ C^\perp, (A \otimes B) \otimes C}}{\vdash A^\perp ⅋ (B^\perp ⅋ C^\perp), (A \otimes B) \otimes C}$$

We shall show below that the maps ρ_A and ρ'_A are indeed $\mathcal{O}(\mathbb{C})$ morphisms and that they form a retraction pair $(\rho'_A, \rho_A) : A \lhd A \otimes I$. First note that $\rho_A : U \to U$. Let $m \in A \otimes I = \{j_1 a k_1 + j_2 b k_2 \,|\, a \in A, b \in I\}^{\perp\perp}$, one computes $\rho_A \bullet m = k_1 m j_1$, now let $p \in A^\perp$, then $j_1 p k_1 (j_1 a k_1 + j_2 b k_2) = j_1 p a k_1$ is nilpotent and hence $j_1 p k_1 \in (A \otimes I)^\perp$, therefore $m \perp j_1 p k_1$ which implies that $k_1 m j_1 \perp p$ and thus $\rho_A \bullet m \in A^{\perp\perp} = A$. Similarly one gets that ρ'_A, λ_A and λ'_A are $\mathcal{O}(\mathbb{C})$ morphisms. Finally one computes that $\rho'_A \rho_A = j_2 j_1 k_1^2 + j_1^2 k_1 k_2 \neq 1_{A \otimes I}$, on the other hand $\rho_A \rho'_A = j_2 k_1 + j_1 k_2 = 1_A$.

We omit the details of the verification of the coherence axioms and the naturality of ρ_A and λ_A in A.

Next we show that $\mathcal{O}(\mathbb{C})$ is symmetric and that the duality defined above on objects can be made into a full and faithful functor on $\mathcal{O}(\mathbb{C})$. Indeed after lengthy computations one gets $s_{B,A} s_{A,B} = j \begin{bmatrix} 0 & j_1 k_1 + j_2 k_2 \\ j_1 k_1 + j_2 k_2 & 0 \end{bmatrix} k$ which is $1_{A \otimes B}$, since $jk = 1_U$.

Given $f : A \to B$ we define $f^\perp : B^\perp \to A^\perp$ as $f^\perp = js(kfj)sk$. $(1_A)^\perp = js(kjskj)sk = jsk = 1_{A^\perp}$ and for $f : A \to B$ and $g : B \to C$, $(gf)^\perp = js(kgfj)sk = jsTr((1 \otimes 1 \otimes s)\tau^{-1}(kfj \otimes kgj)\tau)sk = jTr((1 \otimes 1 \otimes s)\tau^{-1}(k(jskgjsk)j \otimes k(jskfjsk)j)\tau)k = f^\perp g^\perp$ showing that $(-)^\perp$ is a functor. Now let $f, g \in \mathcal{O}(\mathbb{C})(A, B)$ such that $f^\perp = g^\perp$ then $js(kfj)sk = js(kgj)sk$ which implies that $s(kfj)s = s(kgj)s$ and hence $f = g$. Now let $g \in \mathcal{O}(\mathbb{C})(B^\perp, A^\perp)$, let $f = js(kgj)sk$, it is immediate that $f^\perp = g$. Hence $(-)^\perp$ is full and faithful.

We define $A ⅋ B = (A^\perp \otimes B^\perp)^\perp$ and need to show the required isomorphism. Let $f \in \mathcal{O}(\mathbb{C})(A \otimes B, C^\perp)$. Define $\theta(f) = j(1 \otimes j)(k \otimes 1)(kfj)(j \otimes 1)(1 \otimes k)k$ and $\theta'(g) = j(j \otimes 1)(1 \otimes k)(kgj)(1 \otimes j)(k \otimes 1)k$ for $g \in \mathcal{O}(\mathbb{C})(A, B^\perp ⅋ C^\perp)$.

Consider $\theta'(\theta(f)) = j(j\otimes 1)(1\otimes k)(kj(1\otimes j)(k\otimes 1)(kfj)(j\otimes 1)(1\otimes k)kj)(1\otimes j)(k\otimes 1)k = f$. Similarly $\theta(\theta'(g)) = g$ and hence $\theta' = \theta^{-1}$. Thus $\mathcal{O}(\mathbb{C})$ is a $*$-autonomous category. □

Note that in this way we have constructed a model of MLL without units out of a UDC-GoI Situation. We now proceed to construct a model of MELL without units.

Theorem 5.3 *Let* $(\mathbb{C}, T, U)$ *be a UDC-GoI Situation with* $T = (T, \psi, \psi_I)$ *and additional property that*

- $U\otimes U \cong U\ (j,k)$ *and* $TU \cong U\ (u,v)$,
- (T, d', e') *is a comonad,*
- (TA, w'_A, c'_A) *is a commutative comonoid for each* $A \in \mathbb{C}$,
- e'_A *is a map of commutative comonoids,*
- w'_A *and* c'_A *are maps of coalgebras.*

Then there is an endofunctor $(!, \varphi, \varphi_I)$ *on* $\mathcal{O}(\mathbb{C})$ *such that* $(\mathcal{O}(\mathbb{C}), !)$ *is a denotational model of MELL without units.*

Proof.

$! : \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$ is defined as follows. $!(A) = \{uT(a)v \mid a \in A\}^{\perp\perp}$ which clearly is an object in $\mathcal{O}(\mathbb{C})$ and for $f : A \to B$ define

$$!f = j(ue_U \otimes u)\psi^{-1}T((d_U\otimes 1)(kfj)(d'_U\otimes 1))\psi(e'_U v\otimes v)k$$

We show that ! is a functor: $!(1_A) = jsk = 1_{!A}$ which can be shown using a simple matrix calculation and the fact that $e_U T(d_U) = 1_{TU}$ and $T(d'_U)e'_U = 1_{TU}$ and that $uv = 1_U$. Similarly it can be shown that $!(gf) = !g!f$, using the facts above, $vu = 1_{TU}$ and properties of trace.

We next define the monoidal natural transformations:

- $der :! \Longrightarrow Id$ by $der_A :!A \to A = j(ud_U\otimes 1)s(d'_U v\otimes 1)k$. The definition is motivated by:

$$\frac{\vdash A^\perp, A}{\vdash ?A^\perp, A}\ dereliction$$

- $\delta :! \Longrightarrow !!$ by $\delta_A :!A \to !!A = j(ue_U\otimes ue_U)\psi^{-1}T((e_U\otimes 1)\psi^{-1}T((d_U\otimes 1)s(d'_U\otimes 1))\psi(e'_U\otimes 1))\psi(e'_U v\otimes e'_U v)k$, motivated by the proof:

$$\cfrac{\cfrac{\cfrac{\vdash A^\perp, A}{\vdash ?A^\perp, A}\ dereliction}{\vdash ?A^\perp, !A}\ of course}{\vdash ?A^\perp, !!A}\ of course$$

- $weak :! \to \mathcal{K}_I$ by $weak_A :!A \to I = j(uw_U\otimes m)(1_I\otimes 1_I)(w'_U v\otimes n)k = 0_{UU}$, motivated by the proof

$$\frac{\vdash \mathbf{1}}{\vdash ?A^\perp, \mathbf{1}}\ weakening$$

Here $I \lhd U\,(m,n)$, $\mathbf{1}$ is the unit of tensor in MELL and $1_I = 0_{II}$ as I is the zero object in $\mathbb{C}$.

- con $:! \Longrightarrow !\otimes!$ by con_A $:!A \rightarrow !A\otimes!A = j(uc_U \otimes j)(1 \otimes s \otimes 1)(h \otimes h)(1 \otimes s \otimes 1)(c'_U v \otimes k)k$ where $h = (e_U \otimes u)\psi^{-1}T((d_U \otimes 1)s(d'_U \otimes 1))\psi(e'_U \otimes v)$, motivated by the proof

$$\dfrac{\dfrac{\dfrac{\vdash A^{\perp}, A}{\vdash ?A^{\perp}, A}\,der}{\vdash ?A^{\perp}, !A}\,ofcourse \quad \dfrac{\dfrac{\vdash A^{\perp}, A}{\vdash ?A^{\perp}, A}\,der}{\vdash ?A^{\perp}, !A}\,ofcourse}{\dfrac{\vdash ?A^{\perp}, ?A^{\perp}, !A\otimes !A}{\vdash ?A^{\perp}, !A\otimes !A}\,contraction}\,times$$

All the necessary conditions for ! follow from the conditions on T. The well-definedness of monoidal natural transformations above follows from Theorem 4.5. □

6 Double Glueing and Orthogonality

6.1 GoI Construction

In this section we recall Abramsky's $\mathcal{G}$ construction [1]. This is related to the Geometry of Interaction interpretation for MLL in that the composition in the $\mathcal{G}(\mathbb{C})$ uses a version of Girard's execution formula applied to the GoI interpretation of the cut rule. We will describe this construction and then remark that it is equivalent to the *Int* construction of Joyal, Street, and Verity. However, it is more natural to relate $\mathcal{O}(\mathbb{C})$ to $\mathcal{G}(\mathbb{C})$.

Definition 6.1 (The *Geometry of Interaction construction*) Given a traced symmetric monoidal category $\mathbb{C}$ we define a new category $\mathcal{G}(\mathbb{C})$, as follows:

- Objects: Pairs of objects from $\mathbb{C}$, e.g. (A^+, A^-) where A^+ and A^- are objects of $\mathbb{C}$.
- Arrows: An arrow $(A^+, A^-) \xrightarrow{f} (B^+, B^-)$ in $\mathcal{G}(\mathbb{C})$ is $A^+ \otimes B^- \xrightarrow{f} A^- \otimes B^+$ in $\mathbb{C}$. The identity is given by $1_{(A^+,A^-)} = s_{A^+,A^-}$.
- Composition: Composition is given by symmetric feedback. Given $f : (A^+, A^-) \to (B^+, B^-)$ and $g : (B^+, B^-) \to (C^+, C^-)$, $gf : (A^+, A^-) \to (C^+, C^-)$ is given by $gf = Tr^{B^-\otimes B^+}_{A^+\otimes C^-, A^-\otimes C^+}(\beta(f \otimes g)\alpha)$ where α and β are permutations.
- Tensor: $(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-)$ and for $(A^+, A^-) \xrightarrow{f} (B^+, B^-)$ and $(C^+, C^-) \xrightarrow{g} (D^+, D^-)$, $f \otimes g = (1_{A^-} \otimes s_{B^+,C^-} \otimes 1_{D^+})(f \otimes g)(1_{A^+} \otimes s_{C^+,B^-} \otimes 1_{D^-})$ and the tensor unit is (I, I).

Proposition 6.2 *Let $\mathbb{C}$ be a traced symmetric monoidal category. Then $\mathcal{G}(\mathbb{C})$ defined as in Definition 6.1 is a compact closed category. Moreover, $N : \mathbb{C} \to \mathcal{G}(\mathbb{C})$ with $N(A) = (A, I)$ and $N(f) = f$ is a full and faithful embedding.*

Proposition 6.3 ([11]) *Let $\mathbb{C}$ be a traced symmetric monoidal category, then $\mathcal{G}(\mathbb{C}) \cong Int(\mathbb{C})$.*

6.2 Double Glueing

The double glueing construction we recall here is due to Tan and Hyland. Given a compact closed category, this construction produces a $*$-autonomous category which makes tensor and par distinct. The presentation here follows [24] (see also [18]).

Let $\mathbb{C} = (\mathbb{C}, \otimes, I, s, (-)^*)$ be a compact closed category. Let H denote the covariant hom functor $\mathbb{C}(I, -) : \mathbb{C} \to$ **Set** and K denote the contravariant functor $\mathbb{C}(-, I) \cong \mathbb{C}(I, (-)^*) : \mathbb{C}^{op} \to$ **Set**.

Define a new category $\mathbf{G}\mathbb{C}$, *the double glueing category of* $\mathbb{C}$, whose objects are triples $\mathcal{A} = (|\mathcal{A}|, \mathcal{A}_s, \mathcal{A}_t)$ where $|\mathcal{A}|$ is an object of $\mathbb{C}$, $\mathcal{A}_s \subseteq H(|\mathcal{A}|) = \mathbb{C}(I, A)$, is a set of *points* of A, and $\mathcal{A}_t \subseteq K(|\mathcal{A}|) = \mathbb{C}(A, I) \cong \mathbb{C}(I, A^*)$ is a set of *copoints* of A.

A morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathbf{G}\mathbb{C}$ is a morphism $f : |\mathcal{A}| \to |\mathcal{B}|$ in $\mathbb{C}$ such that $Hf : \mathcal{A}_s \to \mathcal{B}_s$ and $Kf : \mathcal{B}_t \to \mathcal{A}_t$. Given $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ in $\mathbf{G}\mathbb{C}$, the composite $gf : |\mathcal{A}| \to |\mathcal{C}|$ is induced by the morphism gf in $\mathbb{C}$. The identity morphism on $\mathcal{A}$ is given by the identity morphism on $|\mathcal{A}|$ in $\mathbb{C}$.

We will denote the underlying object of $\mathcal{A}$ by A, etc. Given objects $\mathcal{A}$ and $\mathcal{B}$ we define the tensor product as follows: $|\ \mathcal{A} \otimes \mathcal{B}\ |= A \otimes B$, $(\mathcal{A} \otimes \mathcal{B})_s = \{\sigma \otimes \tau \mid \sigma \in \mathcal{A}_s, \tau \in \mathcal{B}_s\}$, and $(\mathcal{A} \otimes \mathcal{B})_t = \mathbf{G}\mathbb{C}(\mathcal{A}, \mathcal{B}^\perp)$. where given $\mathcal{A}$, $\mathcal{A}^\perp = (A^*, \mathcal{A}_t, \mathcal{A}_s)$. We define $\mathcal{A} \multimap \mathcal{B} = (\mathcal{A} \otimes \mathcal{B}^\perp)^\perp$ and $\mathcal{A} \parr \mathcal{B} = (\mathcal{A}^\perp \otimes \mathcal{B}^\perp)^\perp$.

Proposition 6.4 (Tan) *For any compact closed category $\mathbb{C}$, $\mathbf{G}\mathbb{C}$ is a *-autonomous category with tensor $\otimes$ as above and unit $\mathbf{1} = (I, \{id_I\}, \mathbb{C}(I, I))$.*

Remark 6.5 Note that $\mathbf{G}\mathbb{C}$ is a non-degenerate categorical model of MLL. That is, the tensor and par products are always distinct. For example, $(I, \emptyset, \emptyset) \otimes (I, \emptyset, \emptyset) = (I, \emptyset, \mathbb{C}(I, I))$ while $(I, \emptyset, \emptyset) \parr (I, \emptyset, \emptyset) = (I, \mathbb{C}(I, I), \emptyset)$.

In a logical setting one can think of an object $\mathcal{A}$ of $\mathbf{G}\mathbb{C}$ as an object A in $\mathbb{C}$ together with a collection of proofs of A (the collection $\mathcal{A}_s$) and a collection of disproofs or refutations of A (the collection $\mathcal{A}_t$.)

Proposition 6.6 *There is a fully faithful monoidal $(-)^\perp$–preserving embedding $F : \mathcal{O}(\mathbb{C}) \to \mathbf{G}(\mathcal{G}\mathbb{C})$.*

Proof. Note that an object in $\mathbf{G}(\mathcal{G}\mathbb{C})$ consists of a triple $((A, B), \mathcal{A}_s, \mathcal{A}_t)$ where A, B are objects in $\mathbb{C}$, $\mathcal{A}_s \subseteq \mathbb{C}(B, A)$ and $\mathcal{A}_t \subseteq \mathbb{C}(A, B)$

The functor F is defined as follows: Given an object $A \in \mathcal{O}(\mathbb{C})$, $F(A) = ((U, U), A, A^\perp)$ and given a morphism $f : A \to B$ in $\mathcal{O}(\mathbb{C})$, $F(f) = kfj$. We shall verify that kfj is indeed a morphism from $((U, U), A, A^\perp)$ to $((U, U), B, B^\perp)$. Clearly $kfj : U \otimes U \to U \otimes U$. Now let $g \in A$; then $\mathcal{G}\mathbb{C}(I, kfj)g = (kfj)g = Tr((1 \otimes g)s(kfj)s) = Tr(s(g \otimes 1)(kfj)s) = f \bullet g$, and we know that

$f \cdot g \in B$ as $g \in A$. Next, let $g \in B^{\perp}$, $\mathcal{G}\mathbb{C}(kfj, I)g = g(kfj) = Tr((1 \otimes g)(kfj)) = Tr(s(g \otimes 1)s(kfj)ss) = f^{\perp} \cdot g$. Recall that $f : A \to B$ and hence $f^{\perp} : B^{\perp} \to A^{\perp}$ and so $f^{\perp} \cdot g \in A^{\perp}$. This verifies that $F(f)$ is a $\mathbf{G}(\mathcal{G}\mathbb{C})$ morphism.

Next we shall verify that F is indeed a functor, $F(1_A) = F(jsk) = k(jsk)j = s = 1_{FA}$. Let $f : A \to B$ and $g : B \to C$, $F(gf) = F(j\, Tr^{U \otimes U}_{U \otimes U, U \otimes U}((1 \otimes 1 \otimes s)\tau^{-1}(kfj \otimes kgj)\tau)k) = Tr^{U \otimes U}_{U \otimes U, U \otimes U}((1 \otimes 1 \otimes s)\tau^{-1}(kfj \otimes kgj)\tau) = F(g)F(f)$. Here $\tau = (1 \otimes 1 \otimes s)(1 \otimes s \otimes 1)$.

Clearly F is injective on objects. Let $f, g : A \to B$ and $F(f) = F(g)$. Then $kfj = kgj$ and so $f = g$. Also given $g : ((U,U), A, A^{\perp}) \to ((U,U), B, B^{\perp})$, $h := jgk : A \to B$ is an $\mathcal{O}(\mathbb{C})$-morphism and $F(h) = g$, hence F is a full and faithful embedding. Observe that $(FA)^{\perp} = ((U,U), A, A^{\perp})^{\perp} = ((U,U), A^{\perp}, A) = F(A^{\perp})$ and given $f : A \to B$, and $F(f^{\perp}) = F(js(kfj)sk) = kjs(kfj)skj = s(kfj)s = (Ff)^{\perp}$. As for the monoidal structure define $\varphi_I : ((I,I), \{1_I\}, \mathbb{C}(I,I)) \to ((U,U), \{1_U\}^{\perp\perp}, \{1_U\}^{\perp})$ by $\varphi_I = 1_U$. Note that $F(A \otimes B) = ((U,U), A \otimes B, (A \otimes B)^{\perp})$ and $F(A) \otimes F(B) = ((U \otimes U, U \otimes U), \mathcal{A}_s, \mathcal{A}_t)$ where $\mathcal{A}_s = \{a \otimes b \,|\, a \in A, b \in B\}$ and $\mathcal{A}_t = \mathbf{G}(\mathcal{G}\mathbb{C})(FA, (FB)^{\perp})$. Define $\varphi_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$ by $\varphi_{A,B} = (1 \otimes s)(s \otimes 1)(j \otimes k)$. $\square$

7 Conclusion and Future Work

In this paper we have used the UDC-GoI Situation to construct a denotational model for MELL without units, thus relating the GoI Semantics to denotational semantics in the case of MELL. While this is fine for "sum" or "particle-style" GoI, the next natural step is to generalize to *any* GoI Situation: this work is currently in progress. The most important aspect of this new work will be the axiomatization of the orthogonality relation (cf. [18]) that will include the nilpotency based definition of Girard as an example. In this way one also hopes to include other categorical implementations of GoI, including "product"-style, like the one by Abramsky and Jagadeesan [4], that do not fit the UDC framework.

In [10], Girard extended the geometry of interaction to the full case, including the additives and constants. He also proved a nilpotency theorem for this semantics and its soundness with respect to a slight modification of familiar sequent calculus in the case of exponential-free conclusions. This too constitutes one of the main parts of our future work and thus construction of denotational models for full LL.

One of the most intriguing questions is full-completeness. While we have given precise connections with the fully complete double-gluing GoI models of MLL in [11], the actual lifting of Hyland-Tan-style full completeness theorems to our setting here appears to be not so straightforward, and is left for future work.

Last but certainly not least, we believe that GoI could be further used in its capacity as a new kind of semantics to analyze PCF and other fragments of functional and imperative languages and be compared to usual denotational and operational semantics through full abstraction theorems.

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