Unifying bisimulation relations
for discrete and continuous systems

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Abstract

The fundamental notion of bisimulation has inspired various notions of system equivalences in concurrency theory. Many notions of bisimulation for various discrete systems have been recently unified in the abstract category theoretical formulation of bisimulation due to Joyal, Nielsen and Winskel. In this paper, we adopt their framework and unify the notions of bisimulation equivalences for discrete, continuous dynamical and control systems. This shows that our equivalence notion is on the right track, but also confirms that abstract bisimulation is general enough to capture equivalence notions in the domain of continuous systems. We believe that the unification of the bisimulation relation for labelled transition systems and dynamical systems under the umbrella of abstract bisimulation, as achieved in this work, is a first step towards a unified approach to modeling of and reasoning about the dynamics of discrete and continuous structures in computer science and control theory.

1   Introduction

In the face of growing complexity of dynamical systems, various methods of complexity reduction are crucial to the analysis and design of such systems. Hence, the problem of equivalence of systems is of great importance to systems and control theory [15].

In the computer science community, and in particular in the field of concurrency theory, researchers have been working on various models and numerous equivalence notions for these models. Among these, process algebras and the notion of bisimulation are by now well established [10]. Category theory has been successfully used to understand and compare the multitude of models for concurrency by Winskel and Nielsen [16]. Related efforts include the categorically inspired framework for comparing models of computation in [9].

In [2], Joyal, Nielsen and Winskel proposed the notion of *span of open maps* in an attempt to understand the various equivalence notions for concurrency in an abstract categorical setting. They also showed that this abstract definition of bisimilarity captures the strong bisimulation relation of Milner [10]. Subsequently in [4] it was shown that abstract bisimilarity can also capture Hennessy's testing equivalences [6], Milner and Sangiorgi's barbed

bisimulation [11] and Larsen and Skou's probabilistic bisimulation [8]. More recently, in [5], Blute et al. formulated a bisimulation relation for Markov processes on Polish spaces in this categorical framework, extending the work of Larsen and Skou. All this evidence further attests to the suitability of this abstract definition as an appropriate venue for formulation of bisimilarity concepts for dynamical, control, and hybrid systems. Other attempts to formulate the notion of bisimulation in categorical language, include the coalgebraic approach of [1, 13].

In this paper we propose a new equivalence relation for dynamical and control systems (see also [12]) that we call bisimulation and further show that this equivalence relation is captured by the abstract bisimulation relation of JNW [2]. This extends the latter abstract framework to the continuous domain. In this paper, our main focus, besides introducing a new equivalence relation for dynamical and control systems, is to establish a unification result for bisimulation of discrete and continuous systems. We postpone the discussion of the important issue of computational aspects of bisimulation for dynamical systems to subsequent work.

Our work also demonstrates the usefulness of a categorical language in transferring important and nontrivial notions between the fields of systems and control theory with a rich analytic and algebraic structure and automata-based models which are the main models in computer science. This is especially important for understanding the correct notions of equivalences for hybrid systems, a subject of our current research.

The rest of the paper is organised as follows: In Section 2, we briefly review the fundamental notion of categories and the abstract formulation of the notion of bisimilarity. Section 3, then provides the main application of this method in concurrency theory and recalls that the abstract bisimilarity captures Milner's strong bisimulation relation. The main theorems and results of our paper are contained in Sections 4 and 5 where we introduce and discuss bisimulation relations for dynamical and control systems respectively. Due to space limitations, we leave the results of Sections 4 and 5 without proof.

2   Abstract Bisimulation

The notion of bisimilarity, as defined in [10], has turned out to be one of the most fundamental notions of operational equivalences in the field of process algebras. This has inspired a great amount of research on various notions of bisimulation for a variety of concurrency models. In order to unify most of these notions, Joyal, Nielson and Winskel gave in [2] an abstract formulation of bisimulation in a category theoretical setting.

Intuitively category theory discusses a class of objects and their relationship to each other. In contrast to set theory where the primary notion is that of membership, in category theory one emphasises the *relations* between the objects. Hence, a category consists of *objects* and *morphisms*. For example, in the category **Set**, the objects are sets, and the morphisms are maps between sets. In the category **Grp** of groups, the objects are groups, and the

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morphisms are group homomorphisms. In the category **Man** of smooth manifolds, the objects are smooth manifolds, and the morphisms are smooth mappings between them. For formal definitions and details related to the categorical notions used in this paper we refer the reader to the standard reference [7].

The approach of [2] introduces a category of models where the objects are the systems in question, and the morphisms are simulations. More precisely, the approach in [2] consists of the following components:

- **Model Category:** The category $\mathcal{M}$ of *models* with objects the systems being studied, and morphisms $f : X \to Y$ in $\mathcal{M}$ should be thought of as a simulation of system X in system Y.

- **Path Category:** The category $\mathcal{P}$, called *the path category*, where $\mathcal{P}$ is a subcategory of $\mathcal{M}$ of path objects and with morphisms expressing how they can be extended.

The path category will serve as an abstract notion of time. Since the path category $\mathcal{P}$ is a subcategory of $\mathcal{M}$ of models, time is thus modeled as a (possibly trivial) dynamical system within the same category $\mathcal{M}$ of models. This allows the unification of notions of time across discrete and continuous domains.

Definition 2.1. A *path* or *trajectory* in an object X of $\mathcal{M}$ is a morphism $p : P \to X$ in $\mathcal{M}$ where P is an object in $\mathcal{P}$.

Let $f : X \to Y$ be a morphism in $\mathcal{M}$, and $p : P \to X$ a path in X, then clearly $f \circ p : P \to Y$ is a path in Y. Note that a path is a morphism in $\mathcal{M}$ and so is the map f and hence $f \circ p$ is a map in $\mathcal{M}$. This is the sense in which Y *simulates* X; any path (trajectory) in X is matched by the path $f \circ p$ in Y.

The abstract notion of bisimulation in [2] demands a slightly stronger version of simulation as follows: Let $m : P \to Q$ be a morphism in $\mathcal{P}$ and let the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{p} & X \\
\downarrow{m} & & \downarrow{f} \\
Q & \xrightarrow{q} & Y
\end{array}
$$

commute in $\mathcal{M}$, i.e., the path $f \circ p$ in Y can be extended via m to a path q in Y. Then there exists $r : Q \to X$ such that in the diagram
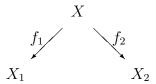
$$
\begin{array}{ccc}
P & \xrightarrow{p} & X \\
\downarrow{m} & \nearrow{r} & \downarrow{f} \\
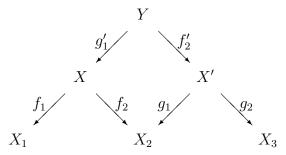Q & \xrightarrow{q} & Y
\end{array}
$$

both triangles commute. Note that this means that the path p can be extended via m to a path r in X which matches q. In this case, we say that $f : X \to Y$ is $\mathcal{P}$-*open*. It can be shown that $\mathcal{P}$-open maps form a subcategory of $\mathcal{M}$.

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Proposition 2.1. *Let $\mathcal{M}$ be a category and $\mathcal{P}$ be the subcategory of path objects. Then, $\mathcal{P}$-open maps in $\mathcal{M}$ form a subcategory of $\mathcal{M}$.*

The definition of $\mathcal{P}$-open maps leads to the notion of $\mathcal{P}$-bisimilarity. We say that objects X_1 and X_2 of $\mathcal{M}$ are $\mathcal{P}$-*bisimilar*, denoted $X_1 \sim_{\mathcal{P}} X_2$ iff there is another object X, and a span of $\mathcal{P}$-open morphisms $f_1 : X \to X_1$ and $f_2 : X \to X_2$ as shown below.

$$
\begin{array}{ccc}
 & X & \\
{}^{f_1}\swarrow & & \searrow^{f_2} \\
X_1 & & X_2
\end{array}
$$

The relation of $\mathcal{P}$-bisimilarity between objects is clearly reflexive (identities are $\mathcal{P}$-open) and symmetric. It is also transitive *provided the model category $\mathcal{M}$ has pullbacks*, due to the fact that pullbacks of $\mathcal{P}$-open morphisms are $\mathcal{P}$-open (see [2] for a proof). Indeed suppose $X_1 \sim_{\mathcal{P}} X_2$ and $X_2 \sim_{\mathcal{P}} X_3$, then $X_1 \sim_{\mathcal{P}} X_3$ as can be seen from the following diagram.

$$
\begin{array}{ccccccc}
 & & & Y & & & \\
 & & {}^{g_1'}\swarrow & & \searrow^{f_2'} & & \\
 & X & & & & X' & \\
{}^{f_1}\swarrow & & \searrow^{f_2} & {}^{g_1}\swarrow & & \searrow^{g_2} & \\
X_1 & & & X_2 & & & X_3
\end{array}
$$

We will see in the upcoming sections below that not all model categories that we consider have pullbacks of all morphisms, in particular the category of smooth manifolds and smooth mappings does not have pullbacks of *all* morphisms. We discuss the solution to this problem in the sections below.

3   Bisimulations of Transition Systems

We briefly illustrate how the framework described in Section 2 results in the usual notion of bisimulation in the sense of Milner [10], for details see [2]. The definitions of transition systems are slightly adapted from [16].

Definition 3.1. A *transition system $T = (S, i, L, \longrightarrow)$* consists of the following:

- A set S of states with a distinguished state $i \in S$ called the *initial state*.

- A set L of labels

- A ternary relation $\longrightarrow \subseteq S \times L \times S$

We form the model category of transition systems $\mathcal{T}$ with objects being transition systems and a morphism $f : T_0 \to T_1$ (with $T_k = (S_k, i_k, L_k, \to_k)$ for $k = 0, 1$) given by $f = (\sigma, \lambda)$ where $\sigma : S_0 \to S_1$ with $\sigma(i_0) = i_1$ and $\lambda : L_0 \to L_1$ a partial function such that

1. $(s, a, s') \in \longrightarrow_0$ and $\lambda(a)$ defined, implies $(\sigma(s), \lambda(a), \sigma(s')) \in \longrightarrow_1$ and

2. $(s, a, s') \in \longrightarrow_0$ and $\lambda(a)$ undefined, implies $\sigma(s) = \sigma(s')$.

In order to discuss the usual bisimilarity of transition systems we need to restrict our model category to the subcategory $\mathcal{T}_L$ of transition systems with the same label set L and morphisms of the form $f = (\sigma, id_L)$ which preserve all the labels. The category $\mathcal{T}_L$ has both binary products and pullbacks.

We define the path category $\mathbf{Bran}_L$ as the full subcategory of $\mathcal{T}_L$ of all synchronisation trees with a single finite branch (possibly empty). Now a path in a transition system T in $\mathcal{T}_L$ is a morphism $p : P \to T$ in $\mathcal{T}_L$, with P an object in $\mathbf{Bran}_L$. Clearly this simply means that we look at the traces of the transition system. The $\mathcal{P}$-open maps in $\mathcal{T}_L$ are now characterized.

Proposition 3.1. *The* $\mathbf{Bran}_L$*-open morphisms of* $\mathcal{T}_L$ *are morphisms* $(\sigma, id_L) : T \to T'$ *with* $T, T' \in \mathcal{T}_L$ *such that:*
If $\sigma(s) \xrightarrow{a} s'$ *in* T'*, then there exists* $u \in S$*,* $s \xrightarrow{a} u$ *in* T *and* $\sigma(u) = s'$*.*

We now recall the strong notion of bisimulation introduced in [10]. Let T_0 and T_1 be two transition systems in $\mathcal{T}_L$.

Definition 3.2. A binary relation $\mathcal{R} \subseteq S_0 \times S_1$ is a *strong bisimulation* if $(s, t) \in \mathcal{R}$ implies, for all $\alpha \in L$:

(i) Whenever $s \xrightarrow{\alpha} s'$ then, there is $t', t \xrightarrow{\alpha} t'$ and $(s', t') \in \mathcal{R}$,

(ii) Whenever $t \xrightarrow{\alpha} t'$ then, there is $s', s \xrightarrow{\alpha} s'$ and $(s', t') \in \mathcal{R}$.

Transition systems T_0 and T_1 are called strongly bisimilar, written $T_0 \sim T_1$, if $(i_0, i_1) \in \mathcal{S}$ for some strong bisimulation relation $\mathcal{S}$. The following theorem, proven in [2], shows that the abstract notion of $\mathbf{Bran}_L$-bisimilarity coincides with the traditional strong notion of bisimulation.

Theorem 3.1 ([2]). *Two transition systems (hence synchronisation trees) over the same labelling set L, are* $\mathbf{Bran}_L$*-bisimilar iff they are* strongly bisimilar *in the sense of Milner [10].*

In the next sections, we consider the notion of $\mathcal{P}$-bisimilarity in the category of dynamical and control systems.

4 Dynamical Systems

A dynamical system or vector field on a manifold M is a smooth section of the tangent bundle on M, that is a smooth map $X : M \to TM$ such that $\pi_M X = id_M$ where $\pi_M : TM \to M$ is the canonical projection of the tangent bundle onto the manifold M.

We proceed to define the model category **Dyn** of dynamical systems. The objects in **Dyn** are dynamical systems $X : M \to TM$ where M is smooth manifold. A morphism in **Dyn** from object $X : M \to TM$ to object $Y : N \to TN$ is a smooth map $f : M \to N$ such that

$$
\begin{array}{ccc}
M & \xrightarrow{\ f\ } & N \\
X \downarrow & & \downarrow Y \\
TM & \xrightarrow{\ Tf\ } & TN
\end{array}
$$

commutes. Thus related systems are said to be *f-related*. The identity morphisms and composition are induced by those in the category **Man** of smooth manifolds and smooth mappings.

We proceed to define the path category $\mathcal{P}$ as the full subcategory of **Dyn** with objects $P : I \to TI$ where I is an open interval of $\mathbb{R}$ and hence a manifold. We also assume that $0 \in I$. Moreover, I is parallelizable (trivializable), that is $TI \cong I \times \mathbb{R}$. Note that a path object represents the differential equation $dx(t)/dt = 1$, with the canonical choice for the section, that is $P(t) = (t, 1)$. This in turn represents a clock running on the interval I.

Definition 4.1. A *path* or *trajectory* in a dynamical system $X : M \to TM$ is a morphism $c : P \to X$ in **Dyn**, where P is an object in $\mathcal{P}$. More explicitly, a path c is a map $c : I \to M$ such that the following diagram commutes.

$$
\begin{array}{ccc}
I & \xrightarrow{\ c\ } & M \\
P \downarrow & & \downarrow X \\
TI & \xrightarrow{\ Tc\ } & TM
\end{array}
$$

This means that a path in X is a smooth map $c : I \to M$ for some open interval I such that $c'(t) = X(c(t))$ for all $t \in I$. Thus, a path in X is just an integral curve in M. Observe that given a path c in X, and $f : X \to Y$, then $f \circ c$ is a path in Y. This is the sense of Y *simulating or over-approximating* X.

The next issue to understand is the meaning of path extension. Suppose $P : I \to TI$ and $Q : J \to TJ$ are objects in $\mathcal{P}$ with I, J open intervals in $\mathbb{R}$ containing the origin, and $m : P \to Q$. Then, m is a smooth map from I to J, such that $m'(t) = 1$ or $m(t) = t - t_0$ for some $t_0 \in I$ and for all $t \in I$.

We now introduce the following notation: let $\phi_X(x_1, x_2)$ denote the predicate that system X evolves from state x_1 to state x_2. Hence, $\phi_X(x_1, x_2)$ is true iff there is an open interval I in $\mathbb{R}$ containing the origin, a path $c : I \to M$ such that $c(0) = x_1$ and $c(t) = x_2$, for some $t \in I$. With this predicate, the characterization of $\mathcal{P}$-open maps is given by the following proposition.

Proposition 4.1. *Given the dynamical systems X on M and Y on N, $f : X \to Y$ is $\mathcal{P}$-open if and only if*

> For any state $x_1 \in M$ of X, if $\phi_Y(f(x_1), y_2)$, then there exists $x_2 \in M$ such that $\phi_X(x_1, x_2)$ where $y_2 = f(x_2)$.

In the particular case where vector fields are *complete*, that is solutions exist for all time, the previous proposition takes the following form.

Proposition 4.2. *Let X and Y on manifolds M and N respectively be complete vector fields. Then any $f : X \to Y$ is $\mathcal{P}$-open.*

Recall that by the general definition in Section 2, two objects X_1 and X_2 in the model category are bisimilar if there is a span of open maps, that is an object X with open maps $f_1 : X \to X_1$ and $f_2 : X \to X_2$. The bisimulation relation has to be an equivalence relation and for that purpose one requires the existence of pullbacks in the underlying model category. As is well known in differential geometry [3], in **Man** arbitrary pullbacks do not exist. Structure needs to be imposed on the maps in order to guarantee that pullbacks exist.

Definition 4.2. Given smooth manifolds M and N, a smooth map $f : M \to N$ and $x \in M$, let $T_x f : T_x M \to T_{f(x)} N$ be the derivative of f. We say that:

(i) f is an *immersion* at x if the map $T_x f$ is injective.

(ii) f is a *submersion* at x if the map $T_x f$ is surjective.

Definition 4.3. Let M, N be smooth manifolds and $f : M \to N$ be a smooth mapping and P be a submanifold of N. The map f is *transversal* over P iff for each $x \in M$ such that $f(x)$ lies in P, the composite

$$T_x(M) \xrightarrow{T_x f} T_{f(x)}(N) \to T_{f(x)}(N)/T_{f(x)}(P)$$

is surjective.

In particular, if for every $x \in M$, $T_x f$ is surjective, that is, if f is a submersion on M, then the composite in the definition above will be surjective and hence every submersion $f : M \to N$ is transversal on every submanifold P of N.

Definition 4.4. Given smooth maps $f : M \to P$ and $g : N \to P$, we say that f and g are transversal if $f \times g : M \times N \to P \times P$ is transversal on the diagonal subset Δ_P of $P \times P$.

Proposition 4.3 ([3]). *Let M and N be smooth manifolds and $f : M \to N$ a smooth map, then $graph(f)$ is a smooth submanifold of $M \times N$.*

Proposition 4.4. *The category **Man** has transversal pullbacks.*

Obviously transversality is a sufficient condition and hence there are other pullbacks in the category **Man**. In view of this proposition we have the following result.

Proposition 4.5. *Pullback of submersions exists in* **Man***. Moreover, the pullback of any submersion is a submersion.*

After all these preliminary results in the category **Man** of manifolds, we can finally get to our desired goal in the category of dynamical systems.

Proposition 4.6. *The category* **Dyn** *has binary products and transversal pullbacks.*

In this case, as we have seen above, we can only guarantee the transversal pullbacks. Hence we modify the definition for $\mathcal{P}$-bisimulation to ensure that it becomes an equivalence relation. That is we require that there be a span of open submersions.

Definition 4.5. We say that two dynamical systems X_1 and X_2 are $\mathcal{P}$-bisimilar if there exists a span of *open submersions* $(Z, f_1 : Z \to X_1, f_2 : Z \to X_2)$.

Note that if there exists an open submersion $f : X_1 \to X_2$, or an open submersion $g : X_2 \to X_1$, then X_1 and X_2 are $\mathcal{P}$-bisimilar. The spans are (X_1, id_{X_1}, f) and (X_2, g, id_{X_2}) respectively. The existence of transversal pullbacks in **Dyn** allows us to show the following result.

Proposition 4.7. *The relation of $\mathcal{P}$-bisimilarity is an equivalence relation on the class of all dynamical systems.*

In order to define bisimilar vector fields, we need a well behaved notion of a relation. The following definition which seems to be new, is inspired by a relevant definition for equivalence relations on manifolds [3, 14].

Definition 4.6. Let M and N be smooth manifolds and $\mathcal{R}$ be a relation from M to N, that is to say $\mathcal{R} \subseteq M \times N$. We say that $\mathcal{R}$ is *regular* iff

- $\mathcal{R}$ is a smooth submanifold of $M \times N$,

- the projection maps $\pi_1 : \mathcal{R} \to M$ and $\pi_2 : \mathcal{R} \to N$ are submersions.

Proposition 4.8. *Let X, Y and Z be smooth manifolds and $\mathcal{R} \subseteq X \times Y$ and $\mathcal{S} \subseteq Y \times Z$ be regular relations. Then $\mathcal{S} \circ \mathcal{R} \subseteq X \times Z$ is a regular relation.*

Definition 4.7. Given two dynamical systems X on M and Y on N, we say that a relation $\mathcal{R} \subseteq M \times N$ is a *bisimulation* relation iff

1. $\mathcal{R}$ is a regular relation,

2. for all $(x, y) \in M \times N$, $(x, y) \in \mathcal{R}$ implies

    - if $\phi_X(x, x')$, there exists $y' \in N$ such that $\phi_Y(y, y')$ and $(x', y') \in \mathcal{R}$
    - if $\phi_Y(y, y')$, there exists $x' \in M$ such that $\phi_X(x, x')$ and $(x', y') \in \mathcal{R}$

We say that two dynamical systems X and Y on manifolds M and N respectively are *bisimilar* if there exists a bisimulation relation $\mathcal{R} \subseteq M \times N$ such that for all $x \in M$ there exists a $y \in N$ with $(x, y) \in \mathcal{R}$ and vice-versa.

Theorem 4.1. *Given dynamical systems X and Y on manifolds M and N respectively, X and Y are bisimilar iff they are $\mathcal{P}$-bisimilar, i.e. $X \sim_{\mathcal{P}} Y$.*

The above theorem shows that the abstract notion of $\mathcal{P}$-bisimilarity coincides with the expected and natural notion of bisimulation for dynamical systems. We now turn our attention to control systems.

5   Bisimulations of Control Systems

We define the model category **Con** as follows. Objects of **Con** are control systems over manifolds, a control system X over a manifold M is given by a pair (U_M, X_M) where $X_M : M \times U_M \to TM$ is a smooth map such that $\pi_M X_M = \pi_1$ with π_M the canonical tangent bundle projection. Here U_M is a smooth manifold called the *input space*. A morphism in **Con** from a control system $X = (U_M, X_M)$ to $Y = (U_N, Y_N)$ is given by a pair (ϕ_1, ϕ_2) of smooth maps with $\phi_1 : M \times U_M \to N \times U_N$ and $\phi_2 : M \to N$, such that

$$
\begin{array}{ccc}
M \times U_M \xrightarrow{\phi_1} N \times U_N & \quad & M \times U_M \xrightarrow{\phi_1} N \times U_N \\
\Big\downarrow{\scriptstyle X_M} \qquad \Big\downarrow{\scriptstyle Y_N} & \quad & \Big\downarrow{\scriptstyle \pi_1} \qquad \Big\downarrow{\scriptstyle \pi_1} \\
TM \xrightarrow{T\phi_2} TN & \quad & M \xrightarrow{\phi_2} N
\end{array}
$$

both commute. Thus related control systems are said to be (ϕ_1, ϕ_2)-related [12]. Note that since π_1 is a surjective map, ϕ_2 is uniquely determined given ϕ_1. The identity morphism $id_X : X \to X$ is given by $id_X = (id_{M \times U}, id_M)$ for any object X in **Con** and given $f : X \to Y$ and $g : Y \to Z$, the composite $gf : X \to Z$ is given by $gf = (g_1 f_1, g_2 f_2)$.

The path category $\mathcal{P}$ is defined as the full subcategory of **Con** with objects control systems (U_I, P_I) where U_I is the singleton space with trivial topology and thus $I \times U_I \cong I$. Hence $P_I : I \to TI$ with $P(t) = (t, 1)$ for all $t \in I$. I is an open interval of $\mathbb{R}$ containing the origin. Thus (I, P_I) is a well defined control system.

Definition 5.1. A *path* in a control system $X = (U_M, X_M)$ is then a morphism $c = (c_1, c_2)$ in **Con** with $c_1 : I \to M \times U_M$ and $c_2 : I \to M$ such that

$$
\begin{array}{ccc}
I \xrightarrow{c_1} M \times U_M & \quad & I \xrightarrow{c_1} M \times U_M \\
\Big\downarrow{\scriptstyle P_I} \qquad \Big\downarrow{\scriptstyle X_M} & \quad & \Big\downarrow{\scriptstyle id_I} \qquad \Big\downarrow{\scriptstyle \pi_1} \\
TI \xrightarrow{Tc_2} TM & \quad & I \xrightarrow{c_2} M
\end{array}
$$

commute.

This means that a path in X is a pair of smooth maps $c_1 : I \to M \times U_M$ and $c_2 : I \to M$ for some open interval I such that $c_2'(t) = X(c_2(t), u(t))$ for all $t \in I$, where $u(t) = \pi_2 c_1(t)$. Let (I, P_I) and (J, Q_J) be two path objects in $\mathcal{P}$ and $m = (m_1, m_2) : P \to Q$ be a path extension. Then from the diagram on the right above we get that $m_1 = m_2 : I \to J$ and then the diagram on the left coincides with the condition we had for dynamical systems. Thus a path extension $m = (m_1, m_2)$ is of the form $m_1 = m_2 : I \to J$, $m_1(t) = t - t_0$ for $t_0 \in I$ and for all $t \in I$.

Proposition 5.1. *The category* **Con** *has binary products and transversal pullbacks.*

We introduce the following notation: let $\phi_X(x_1, x_2)$ denote the predicate that system $X = (U_M, X_M)$ evolves from state x_1 to state x_2 under some input in U_M. Hence, $\phi_X(x_1, x_2)$ is true iff there is an open interval I of $\mathbb{R}$ containing the origin, a path $c = (c_1, c_2)$ such that $c_2(0) = x_1$ and $c_2(t) = x_2$, for some $t \in I$. The input deriving the system is given by $\pi_2 c_1 : I \to U_M$. Similarly to the case of dynamical systems, we characterize the $\mathcal{P}$-open maps as follows.

Proposition 5.2. *Given the control systems $X = (M, X_M)$ and $Y = (N, Y_N)$, $f : X \to Y$ is $\mathcal{P}$-open iff*

> *For any state $x_1 \in M$ of X, if $\phi_Y(f(x_1), y_2)$, then there exists $x_2 \in M$ such that $\phi_X(x_1, x_2)$ where $y_2 = f(x_2)$.*

Definition 5.2. We say that two control systems X_1 and X_2 are $\mathcal{P}$-bisimilar if there exists a span of *open submersions* $(Z, f_1 : Z \to X_1, f_2 : Z \to X_2)$.

Proposition 5.3. *The relation of $\mathcal{P}$-bisimilarity is an equivalence relation on the class of all control systems.*

We define the bisimulation relation for control systems, similarly to the case of dynamical systems.

Definition 5.3. Given two control systems $X = (U_M, X_M)$ and $Y = (U_N, Y_N)$, we say that a relation $\mathcal{R} \subseteq M \times N$ is a *bisimulation* relation iff

1. $\mathcal{R}$ is a regular relation,

2. for all $(x, y) \in M \times N$, $(x, y) \in \mathcal{R}$ implies

    - if $\phi_X(x, x')$, there exists $y' \in N$ such that $\phi_Y(y, y')$ and $(x', y') \in \mathcal{R}$
    - if $\phi_Y(y, y')$, there exists $x' \in M$ such that $\phi_X(x, x')$ and $(x', y') \in \mathcal{R}$

We say that two control systems X and Y as above are *bisimilar* if there exists a bisimulation relation $\mathcal{R} \subseteq M \times N$ such that for all $x \in M$ there exists a $y \in N$ with $(x, y) \in \mathcal{R}$ and vice-versa.

Theorem 5.1. *Given control systems $X = (U_M, X_M)$ and $Y = (U_N, Y_N)$, X and Y are bisimilar if and only if they are $\mathcal{P}$-bisimilar, i.e. $X \sim_{\mathcal{P}} Y$.*

The above theorem, shows how the categorical notion of bisimulation described in Section 2, also captures the expected notion of bisimulation for control systems.

6   Conclusions and Future Work

In this paper we propose a new equivalence notion for dynamical and control systems that we called bisimulation, we also proved that this definition is captured in both cases (dynamical and control systems) by the abstract bisimulation of JNW. As a natural extension of the present work, currently we are studying the formulation of bisimulation relation for hybrid dynamical systems and algebraic characterisations for bisimilarity that can lead to efficient computational methods. The abstract bisimilarity is also well connected with logic and game characterisations of bisimulation and presheaf semantics in the case of concurrency models [17]. Currently ongoing work includes the study and development of similar connections for dynamical, control and hybrid systems. In this way, we hope to get a natural specification logic for the description of properties of such systems. We hope that the present work can provide a framework general enough, thanks to category theoretical tools, in which to study a unified approach to the dynamics of discrete and continuous systems.

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